Noncommutative Bianchi Type II Quantum Cosmology

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Abstract In this paper we present the noncommutative Bianchi Class A cosmological models coupled to barotropic perfect fluid. The commutative and noncommutative quantum solution to the Wheeler–DeWitt equation for any factor ordering, to the anisotropic Bianchi type II cosmological model are found, using a stiff fluid ($\gamma = 1$). In our toy model, we introduce noncommutative scale factors, is say, we consider that all minisuperspace variables q^i does not commute, so the simplectic structure was modified.

1 Introduction

In the last few years there has been several attempts to study the possible effects of noncommutativity in the cosmological scenario. In particular, Compean et al. [1–3] avoid the difficult technicalities of analyzing noncommutative cosmological models, when these are derived from the full noncommutative theory of gravity, see by example [4–11], their proposal is to introduce the effects of noncommutativity in quantum cosmology including one

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θ parameter, deforming the minisuperspace. This deformation is achieved through a Moyal deformation of the Wheeler–DeWitt (WDW) equation, this is similar to noncommutative quantum mechanics [11, 12]. Some work has been done in this direction, for example in [13] the authors study the implications of noncommutative geometry in minisuperspace variables for a FRW universe with a conformally coupled scalar field, using the Bohmian formalism of quantum trajectories [14] also in [15] a noncommutative deformation of a scalar field coupled to scalar-tensor type gravity was considered.

The aim of this paper is to build a noncommutative scenario for the Bianchi Class A cosmological models coupled with matter, we introduce noncommutative scale factors in Bianchi cosmologies and compare their solutions to that of the commutative case at quantum level. Thus, the noncommutativity is considered between all variables of the minisuperspace (not in its momenta) with three θ_i parameters, $[q^i, q^j] = i\theta^{ij}$, usually this is made by the Moyal star product $\star$, however, we use one transformation between the noncommutative and the commutative variables, resulting in a shifting, and it is well known that this correspondence, implies changes in the potential term of the WDW equation with a shifting in the variables [1, 15]. These transformation are not the most general possible to define a noncommutative fields, see for example Carmona et al. [16], where the authors give the generalization the noncommutative harmonic oscillator constructions and shown the most linear transformation between the coordinates and momenta. In Vakiri et al. [17] one procedure similar is applied to Bianchi I without matter.

On the other hand, the inclusion of matter to homogeneous cosmologies has been treated with scalar fields in order to study different scenarios, inflation, dark matter, dark energy. However, since 1972 year, the problem of the appropriate sources of matter pointed in [18, 19], and their corresponding Lagrangian for each of them has not been solved. One of the sources that is usually considered, is the perfect fluid, in this paper we consider this source as a first approximation in the noncommutative quantization program, in particular for the Bianchi type II cosmological model is considered, as a toy model, with the idea to apply this procedure in a future to all Bianchi Class A models.

The paper is then organized as follows. In Sect. 2, we obtain the WDW equation including the barotropic matter contribution. In Sect. 3 we present the commutative quantum solutions for the cosmological Bianchi type II, and stiff fluid as the matter source, Sect. 4 is devoted to the noncommutative quantum model and the noncommutative quantum solutions are presented. Final remarks are presented in Sect. 5.

2 The Wheeler–DeWitt Equation

Let us start by recalling the canonical formulation of the ADM formalism for the diagonal Bianchi Class A models. The metrics have the form

$$ds^2 = -(N^2 - N^j N_j)dt^2 + e^{2\Omega(t)}e^{2\beta_{ij}(t)}\omega^i\omega^j, \tag{1}$$

where N and N_i are the lapse and shift functions, respectively, $\Omega(t)$ is a scalar and $\beta_{ij}(t)$ a 3×3 diagonal matrix, $\beta_{ij} = \text{diag}(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+)$, ω^i are one-forms that characterize each cosmological Bianchi type model, and that obey $d\omega^i = \frac{1}{2}C^i_{jk}\omega^j \wedge \omega^k$, C^i_{jk} the structure constants of the corresponding invariance group [19].

For the Bianchi type II, has the form

$$
\begin{aligned}
ds^2 = {} & -N^2 dt^2 + e^{2\Omega} e^{-4\beta_+} dx^2 + e^{2\Omega} e^{2\beta_+ + 2\sqrt{3}\beta_-} dy^2 \\
& + e^{2\Omega}[x^2 e^{2\beta_+ + 2\sqrt{3}\beta_-} + e^{2\beta_+ - 2\sqrt{3}\beta_-}] dz^2 \\
& - x e^{2\Omega} e^{2\beta_+ + 2\sqrt{3}\beta_-} dy dz - x e^{2\Omega} e^{2\beta_+ + 2\sqrt{3}\beta_-} dz dy.
\end{aligned}
\tag{2}
$$

The Lagrangian is given by

$$
L = 6e^{3\Omega}\left[\frac{\dot{N}\dot{\Omega}}{N^2} - \frac{\ddot{\Omega}}{N} - 2\frac{\dot{\Omega}^2}{N} - \frac{\dot{\beta}_+^2}{N} - \frac{\dot{\beta}_-^2}{N} + \frac{1}{12}Ne^{-2\Omega + 4\beta_+ + 4\sqrt{3}\beta_-} + \frac{8}{3}\pi GN\rho\right],
\tag{3}
$$

where we used a perfect fluid as the matter content in a comoving frame, see [18, 20]. When we rewrite this equation in a canonical way and using the solution to $T^{\mu\nu}{}_{;\nu} = 0$, we arrive to the Hamiltonian function

$$
H = \frac{e^{-3\Omega}}{24}[-P_\Omega^2 + P_+^2 + P_-^2 + 12e^{4(\Omega + \beta_+ + \sqrt{3}\beta_-)} + 384\pi GM_\gamma e^{-3(\gamma-1)\Omega}] = 0,
$$

and we assumed a barotropic state equation $p = \gamma\rho$, where the parameter, $-1 \leq \gamma \leq 1$.

By replacing P_{q^μ} by $-i\partial_{q^\mu}$ in (2), with $q^\mu = (\Omega, \beta_+, \beta_-)$, we get the WDW equation. Following Hartle–Hawking [21] we introduce a semi-general factor ordering, this gives

$$
\left[\frac{\partial^2}{\partial\Omega^2} - \frac{\partial^2}{\partial\beta_+^2} - \frac{\partial^2}{\partial\beta_-^2} + q\frac{\partial}{\partial\Omega} + 12e^{4(\Omega + \beta_+ + \sqrt{3}\beta_-)} + 384\pi GM_\gamma e^{-3(\gamma-1)\Omega}\right]\Psi = 0,
\tag{4}
$$

that is equivalent to

$$
\Box\Psi - q\frac{\partial\Psi}{\partial\Omega} - U(\Omega, \gamma) = 0,
\tag{5}
$$

where $\Box$ is the d'Alambertian in three dimensions with signature $(-, +, +)$, and the potential function $U(\Omega, \gamma) = 12e^{4(\Omega + \beta_+ + \sqrt{3}\beta_-)} + 384\pi GM_\gamma e^{-3(\gamma-1)\Omega}$.

3 Commutative Cosmological Quantum Solutions, for $\gamma = 1$

Now we consider a stiff fluid $\gamma = 1$ as the matter content for our model, for the anisotropic commutative model. We simplify (5), introducing the particular transformation between the coordinates

$$
\xi = \Omega + \beta_+ + \sqrt{3}\beta_-, \qquad \kappa = \Omega + \frac{\sqrt{3}}{3}\beta_-, \qquad \lambda = \Omega - 2\beta_+ + \sqrt{3}\beta_-,
$$

having

$$
q\frac{\partial\Psi}{\partial\xi} - 3\frac{\partial^2\Psi}{\partial\xi^2} + (12e^{4\xi} + 384\pi GM_0)\Psi + \frac{2}{3}\frac{\partial^2\Psi}{\partial\kappa^2} + q\frac{\partial\Psi}{\partial\kappa} - 6\frac{\partial^2\Psi}{\partial\lambda^2} + q\frac{\partial\Psi}{\partial\lambda} = 0.
\tag{6}
$$

Using the separation variables method with $\Psi(\xi, \kappa, \lambda) = X(\xi)Y(\kappa)Z(\lambda)$ and substituting in (6), we obtain the set of differential equations

$$
-\frac{3}{X}\frac{d^2X}{d\xi^2} + \frac{q}{X}\frac{dX}{d\xi} + (12e^{4\xi} + C_0) = -12\mu^2,
\tag{7}
$$

$$\frac{2}{3Y}\frac{d^2Y}{d\kappa^2} + \frac{q}{Y}\frac{dY}{d\kappa} = -\frac{2}{3}b_1^2, \tag{8}$$

$$-\frac{6}{Z}\frac{d^2Z}{d\lambda^2} + \frac{q}{Z}\frac{dY}{d\lambda} = 6c_1^2, \tag{9}$$

with $C_0 = 384\pi GM_1$, and for simplicity we choose the constants μ, b_1, c_1 in such away that satisfying the relation between them as $\frac{2}{3}b_1^2 - 6c_1^2 = -12\mu^2$.

Introducing the change of variables $z = e^{4\xi}$ in (7), we get an ordinary Bessel differential equation for z

$$z^2\frac{d^2X}{dz^2} - \left(1 + \frac{q}{12}\right)z\frac{dX}{dz} - \left(\frac{z}{4} + b_0\right)X = 0, \tag{10}$$

where $b_0 = \frac{1}{48}(C_0 + 12\mu^2)$, so the physical solution is

$$X(\xi)_\nu = e^{q\xi}\,K_\nu(e^{2\xi}), \tag{11}$$

with $\nu = \frac{1}{2}\sqrt{4\mu^2 + 128\pi GM_1 + q^2}$.

The solutions for the other two equations are

$$Y(\kappa) = e^{-\frac{3q}{4}\kappa}[A_1e^{ip_1\kappa} + A_2e^{-ip_1\kappa}], \tag{12}$$

$$Z(\lambda) = e^{\frac{q}{12}\lambda}[B_1e^{ip_2\lambda} + B_2e^{-ip_2\lambda}], \tag{13}$$

where $p_1 = \sqrt{b_1^2 - \frac{9q^2}{16}}$ and $p_2 = \sqrt{c_1^2 - \frac{q^2}{144}}$.

Finally, we find the wave function

$$\Psi_{C_\nu} = e^{4\xi + q(\xi - \frac{3}{4}\kappa + \frac{1}{12}\lambda)}\,K_\nu(e^{2\xi})[A_1e^{ip_1\kappa} + A_2e^{-ip_1\kappa}][B_1e^{ip_2\lambda} + B_2e^{-ip_2\lambda}]. \tag{14}$$

For the factor ordering $q = 0$, the solution is simplified.

4 Noncommutative Cosmological Quantum Solutions

Finally we can proceed to the noncommutative model, actually we will consider, that the minisuperspace variables q^i do not commute, so the simplectic structure is modified as follows

$$[q^i, q^j] = i\theta^{ij}, \qquad [P_i, P_j] = 0, \qquad [q^i, P_j] = i\delta^i_j, \quad q^i = (\Omega, \beta_+, \beta_-), \tag{15}$$

in particular, we choose the following representation

$$[\Omega, \beta_+] = i\theta_1, \qquad [\Omega, \beta_-] = i\theta_2, \qquad [\beta_-, \beta_+] = i\theta_3 \tag{16}$$

where the θ_i parameters are a measure of the noncommutativity between the minisuperspace variables. The commutation relation (15) or (16) are not the most general ones to define a noncommutative field.

It is well known, that this non-commutativity can be formulated in term of noncommutative minisuperspace functions with the Moyal star product $\star$ of functions

$$f(\Omega, \beta_+) \star g(\Omega, \beta_+) = f(\Omega, \beta_+)\, e^{i\frac{\theta_1}{2}(\overleftarrow{\partial}_\Omega\overrightarrow{\partial}_{\beta_+} - \overleftarrow{\partial}_{\beta_+}\overrightarrow{\partial}_\Omega)}\, g(\Omega, \beta_+),$$

$$f(\Omega, \beta_-) \star g(\Omega, \beta_-) = f(\Omega, \beta_-)\, e^{i\frac{\theta_2}{2}(\overleftarrow{\partial}_\Omega \overrightarrow{\partial}_{\beta_-} - \overleftarrow{\partial}_{\beta_-} \overrightarrow{\partial}_\Omega)} g(\Omega, \beta_-), \qquad (17)$$

$$f(\beta_-, \beta_+) \star g(\beta_-, \beta_+) = f(\beta_-, \beta_+)\, e^{i\frac{\theta_3}{2}(\overleftarrow{\partial}_{\beta_-} \overrightarrow{\partial}_{\beta_+} - \overleftarrow{\partial}_{\beta_+} \overrightarrow{\partial}_{\beta_-})} g(\beta_-, \beta_+).$$

On the other hand, it is well known that this correspondence, implies changes in the potential term of the WDW equation with a shift in the variables [1, 15], and one possibility for recover (16), and maintaining the old relations to the noncommutative fields, it become

$$\Omega_{nc} \rightarrow \Omega + \frac{\theta_1}{2} P_+ + \frac{\theta_2}{2} P_-, \qquad \beta_{-nc} \rightarrow \beta_- - \frac{\theta_2}{2} P_\Omega + \frac{\theta_3}{2} P_+,$$

$$\beta_{+nc} \rightarrow \beta_+ - \frac{\theta_1}{2} P_\Omega - \frac{\theta_3}{2} P_-, \qquad (18)$$

but, these shifting modified the potential term as

$$U(\Omega, \beta_\pm, \theta_i) = 12 e^{4[\Omega + \beta_+ + \sqrt{3}\beta_- + \frac{i\theta_1}{2}(\frac{\partial}{\partial\Omega} - \frac{\partial}{\partial\beta_+}) + \frac{i\theta_2}{2}(\sqrt{3}\frac{\partial}{\partial\Omega} - \frac{\partial}{\partial\beta_-}) - \frac{i\theta_3}{2}(\sqrt{3}\frac{\partial}{\partial\beta_+} - \frac{\partial}{\partial\beta_-})]} + C_0. \quad (19)$$

Now we can construct the noncommutative WDW equation (NCWDW)

$$\left[\frac{\partial^2}{\partial\Omega^2} - \frac{\partial^2}{\partial\beta_+^2} - \frac{\partial^2}{\partial\beta_-^2} + q\frac{\partial}{\partial\Omega} + C_0 \right.$$
$$\left. + 12 e^{4[\Omega + \beta_+ + \sqrt{3}\beta_- + \frac{i\theta_1}{2}(\frac{\partial}{\partial\Omega} - \frac{\partial}{\partial\beta_+}) + \frac{i\theta_2}{2}(\sqrt{3}\frac{\partial}{\partial\Omega} - \frac{\partial}{\partial\beta_-}) - \frac{i\theta_3}{2}(\sqrt{3}\frac{\partial}{\partial\beta_+} - \frac{\partial}{\partial\beta_-})]} \right] \Psi = 0. \quad (20)$$

Using the generalized Baker–Campbell–Hausdorff formula [22, 23]

$$e^{\eta(\hat{A}+\hat{B})} = e^{-\eta^2[\hat{A},\hat{B}]}\, e^{\eta\hat{A}}\, e^{\eta\hat{B}} \qquad (21)$$

and the relation between the variables (16), we obtain in the new variables (ξ, κ, λ)

$$q\frac{\partial\Psi}{\partial\xi} - 3\frac{\partial^2\Psi}{\partial\xi^2} + (12 e^{4\xi} e^{2i\theta_1 \frac{\partial}{\partial\kappa}} e^{6i\theta_1 \frac{\partial}{\partial\lambda}} e^{\frac{4\sqrt{3}}{3}i\theta_2 \frac{\partial}{\partial\kappa}} e^{\frac{2\sqrt{3}}{3}i\theta_3 \frac{\partial}{\partial\kappa}} e^{6\sqrt{3}i\theta_3 \frac{\partial}{\partial\lambda}} + C_0)\Psi$$

$$+ \frac{2}{3}\frac{\partial^2\Psi}{\partial\kappa^2} + q\frac{\partial\Psi}{\partial\kappa} - 6\frac{\partial^2\Psi}{\partial\lambda^2} + q\frac{\partial\Psi}{\partial\lambda} = 0. \qquad (22)$$

Now we can look for solutions, using the particular ansatz $\Psi = X(\xi)e^{(-\frac{3q}{4} \pm ip_1)\kappa} e^{(\frac{q}{12} \pm ip_2)\lambda}$, on (22), and taking in account that $e^{i\theta \frac{\partial}{\partial x}} e^{\eta x} \equiv e^{i\eta\theta} e^{\eta x}$ yields the equation for the function $X(\xi)$

$$-3\frac{d^2X}{d\xi^2} + q\frac{dX}{d\xi} + (12 e^{f(\theta_i)} e^{4\xi} + C_0 + 12\mu^2)X = 0, \qquad (23)$$

where the function $f(\theta_i)$ have the following structure

$$f(\theta_i) = \theta_1[-iq \mp (2p_1 + 6p_2)] + \theta_2\left(-i\sqrt{3}q \mp \frac{4\sqrt{3}}{3}p_1\right) \mp \theta_3\sqrt{3}\left(\frac{2}{3}p_1 + 6p_2\right), \quad (24)$$

and the constant b_1, c_1 and μ maintain the same relation as in the commutative case. We can see that (7) and (23) have the same structure, then the corresponding solutions to (23)

become the modified Bessel function

$$X_{\mathrm{NC}_\nu}(z) = e^{q\xi} K_\nu(e^{f(\theta_i)} e^{2\xi}), \tag{25}$$

Finally, we arrive to the noncommutative wave function

$$\Psi_{\mathrm{NC}_\nu} = e^{4\xi + q(\xi - \frac{3}{4}\kappa + \frac{1}{12}\lambda)} K_\nu(e^{f(\theta)} e^{2\xi}) e^{\pm i p_1 \kappa} e^{\pm i p_2 \lambda}. \tag{26}$$

We can remark that the noncommutative correction the Bessel function gives the commutative solutions when the parameters $\theta_i \to 0$ or for particular choose between the parameters in such a way that $f(\theta_i) = 0$.

5 Conclusions

In this work we introduce noncommutative scale factors, is say, consider that all minisuperspace variables q^i does not commute, so the simplectic structure was modified. We have extended the non commutativity between the minisuperspace variables (fields) to three parameters as

$$[\Omega, \beta_+] = i\theta_1, \qquad [\Omega, \beta_-] = i\theta_2, \qquad [\beta_-, \beta_+] = i\theta_3. \tag{27}$$

Also we present one particular transformation, (19), where these relations are hold, but does not unique, and are not the most general ones to define a noncommutative fields. Also we have found the commutative and noncommutative quantum solutions for the Bianchi type II cosmological model, considering a scenario dominated by stiff fluid. The noncommutative quantum solution have different structure with respect to the commutative solutions, modified by the function $f(\theta_i)$, (24). The commutative solutions is recover if the parameters $\theta_i \to 0$ or for particular choose between the parameters in such a way that $f(\theta_i) = 0$. This procedure will be applied to other Bianchi Class A models, and the results will be reported elsewhere.

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